Role of Buoyancy and the Boussinesq Approximation in Horizontal Boundary Layers

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The effect of buoyancy on the shear and heat transfer in a horizontal boundary layer is studied for a fluid which has a linear density, viscosity and conductivity variation with temperature. It is shown that the problem admits a similarity solution for arbitrary buoyancy when the external stream is accelerated as the one-fifth power of the distance from the leading edge. This case is solved numerically for both the complete and the Boussinesq approximation to the boundary-layer problem. It is shown that the Boussinesq approximation gives quite accurately a correction to the shear and heat-transfer coefficient due to buoyancy, except in the case of cooled flows where the boundary layer tends to a buoyancy-induced separation. A perturbation expansion to the Boussinesq equations for small buoyancy which admits a family of self-similar solutions for flows accelerated or decelerated as powers of the distance is derived and the coupling between pressure gradient and (small) buoyancy effects is discussed.

Nomenclature

c_p = specific heat at constant pressure
C_f = friction coefficient, see Eq. (30a)
C_h = heat-transfer coefficient, see Eq. (30b)
f = dimensionless streamfunction for non-Boussinesq equations
F = Froude number; also dimensionless streamfunction for Boussinesq equations
g = dimensionless temperature for Boussinesq equations
g_i = dimensionless streamfunction, see Eq. (36a)
h = dimensionless temperature for Boussinesq equations
k = thermal conductivity
L = characteristic horizontal dimension
M = Mach number
p = pressure
P = Prandtl number
R = Reynolds number
s_i = dimensionless temperature, see Eq. (36b)
t = dimensional temperature
T = dimensionless temperature
u = dimensionless horizontal velocity
v_1 = horizontal velocity
v_3 = vertical velocity
w = dimensionless vertical velocity
x = dimensionless horizontal coordinate
x_1 = horizontal coordinate
x_3 = vertical coordinate
z = dimensionless vertical coordinate
α = dimensionless thermal expansion coefficient ($\alpha_0 t_0$)
Θ = dimensionless temperature potential ($t_w - t_0/t_0$)
δ = buoyancy parameter $(FR^{1/2})^{-1}$
π = dimensioness pressure
η = similarity variable
ρ = mass density
μ = dynamic viscosity
Ψ = streamfunction

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Introduction

WHEN the temperature differences across a boundary layer are sufficiently large, buoyancy forces due to density stratification within the boundary layer may not be negligible. Buoyancy effects vary as the product of the temperature difference, the coefficient of thermal expansion of the fluid, and a scale represented by the inverse product of the Froude and the square root of Reynolds numbers. Their importance in low-speed ($U^{-3/2}$) large-scale ($L^{1/2}$) flows such as occur in geophysical fluid dynamics is well-recognized. They can also become important in engineering applications involving intense transfer of heat to liquid coolants (e.g., nuclear reactors, rotating elements subjected to high centrifugal acceleration).

The effect of buoyancy on laminar forced convection over a horizontal surface has been investigated by Mori,[1] Sparrow, and Minkowycz,[2] and Hauptmann.[3] The work of Mori contained an error in the sign of the buoyancy term which was corrected by Sparrow and Minkowycz. They showed that buoyancy increases the skin friction and heat transfer on the upper surface of a heated plate and conversely for a cooled plate. Hauptmann obtained the same results by means of an approximate integral technique, whereas Mori and Sparrow and Minkowycz derived similarity equations which were integrated numerically.

All of the above mentioned analyses were based on the Boussinesq approximation for unaccelerated flow and involved an expansion of the dependent variables about the Blasius solution, thereby relegating the influence of buoyancy to a secondary role. The Boussinesq approximation neglects the effect of density variations on the inertia terms but retains the buoyant body force. Since both the convective and the body force terms depend on the thermal potential across the boundary layer, this potential must be large for the existence of substantial buoyancy effects, whereas the foregoing analyses are limited to small perturbations with respect to buoyancy. An evaluation of the full effect of buoyancy on the boundary layer on a horizontal surface and a comparison with results obtained using the Boussinesq approximation is of considerable interest.

In the following section we show that one self-similar solution exists for the general laminar boundary-layer equations when fluid properties are functions of temperature only. This self-similar case requires an external flow accelerated as $x^{1/5}$. Numerical results obtained from an integration of these equations allow an evaluation of both the effect of the Boussinesq approximation and the effect of buoyancy. A perturbation expansion valid for small buoyancy for arbitrary acceleration is presented and applied to the Boussinesq equations in order to evaluate the coupling of buoyancy and free-stream acceleration.

Analysis for Arbitrary Buoyancy

The equations of motion in the boundary layer for two-dimensional flow of a fluid with a constant specific heat, neglecting dissipation, are given by

$$(\rho u)_x + (\rho w)_z = 0 \tag{1}$$

$$\rho(uu_x + wu_z) = -\pi_x + (\mu u_z)_z \tag{2}$$

$$0 = \pi_z + \delta \tag{3}$$

$$\rho(uT_x + wT_z) = (1/P)(kT_z)_z \tag{4}$$

We shall postulate a phenomenological description of the fluid of the form

$$\rho = \rho(T) \tag{5a}$$

$$\mu = \mu(T) \tag{5b}$$

$$k = k(T) \tag{5c}$$

The notation is standard, and all variables have been made dimensionless by means of the following definitions

$$\begin{gathered} u = v_1/U_o, \quad w = (v_3/U_o)R^{1/2}, \quad x = x_1/L, \\ z = (x_3/L)R^{1/2}, \quad \pi = (p - p_o)/\rho_o U_o^2, \\ \rho = \rho^*/\rho_o, \quad T = (t - t_o)/(t_w - t_o), \\ \mu = \mu^*/\mu_o, \quad \text{and } k = k^*/k_o \end{gathered} \tag{6}$$

The dimensionless parameters characterizing the problem are the Prandtl number $P = (\mu_o c_{po}/k_o)$, the Reynolds number $R = (\rho_o U_o L/\mu_o)$, and the Froude number $F = (U_o^2/gL)$. The parameter δ appearing in the vertical momentum equation is defined as

$$\delta = (FR^{1/2})^{-1} \tag{7}$$

The equation of state (5a) represents the behavior of a large group of liquids. It can also be viewed as an approximation to the equation of state of an ideal gas when pressure variations (dynamic or hydrostatic) are small in comparison with the product of the coefficient of thermal expansion of the gas and the temperature difference across the boundary layer. This means that the analysis applies then to low-speed flow ($M_o^2 \ll 1$, M_o = Mach no.) which is consistent with the neglect of compressional and frictional heating. The assumption of a constant specific heat could be removed by formulating the problem in terms of enthalpy and replacing the thermal conductivity by the ratio (k/c_p).

Using the equation of state (5a), the energy Eq. (4) can be rewritten as

$$\rho(u\rho_x + w\rho_z) = 1/P\{k\rho_{zz} + k(\rho_z)^2[(d^2T/d\rho^2)/(dT/d\rho)] + (\rho_z)^2(dk/dT)dT/d\rho\} \tag{8}$$

If we now multiply Eq. (2) by the density ρ and Eq. (8) by the velocity u and add the resulting equations, the horizontal momentum equation becomes

$$\begin{aligned} \rho u(\rho u)_x + \rho w(\rho u)_z + \rho\pi_x = {} & \mu\left\{(\rho\mu)_{zz} - 2\rho_z\left(\frac{u}{\rho}\right)_z + \frac{k/\mu - P}{P}\, u\, \frac{\rho_{zz}}{\rho}\right\} + \\ & \frac{d\mu}{dT}\frac{dT}{d\rho}\left\{\rho_z u_z + \frac{1}{P}\left(\frac{dk/dT}{d\mu/dt} - P\right) u\, \frac{(\rho_z)^2}{\rho}\right\} + \\ & \frac{k}{P}\left(\frac{d^2T/d\rho^2}{dT/d\rho}\right) u\, \frac{(\rho_z)^2}{\rho} \end{aligned} \tag{9}$$

Defining a streamfunction by

$$\psi_z = \rho u \quad \text{and} \quad \psi_x = -\rho w \tag{10}$$

and eliminating the pressure π between Eqs. (3) and (9) yields the following two equations for the two dependent variables ρ and ψ

$$\begin{aligned} -\delta\rho_x = \frac{\partial}{\partial z}\Bigg\{\frac{1}{\rho}\Bigg[& -J_{(\psi_z,\psi)} + \mu\left\{\psi_{zzz} - 2\rho_z\left(\frac{\psi_z}{\rho}\right)_z + \frac{k/\mu - P}{P}\,\psi_z\,\frac{\rho_{zz}}{\rho}\right\} + \frac{d\mu}{dT}\frac{dT}{d\rho}\left\{\rho_z\psi_{zz} + \frac{1}{P}\left(\frac{dk/dT}{d\mu/dT} - P\right)\psi_z \times \right. \\ & \left.\frac{(\rho_z)^2}{\rho}\right\} + \frac{k}{P}\,\frac{d^2T/d\rho^2}{dT/d\rho}\,\psi_z\,\frac{(\rho_z)^2}{\rho}\Bigg]\Bigg\} \end{aligned} \tag{11}$$

$$J_{(\rho,\psi)} = \frac{k}{P}\left[\rho_{zz} + \frac{d^2T/d\rho^2}{dT/d\rho}(\rho_z)^2\right] + \frac{1}{P}\frac{dk}{dT}\frac{dT}{d\rho}(\rho_z)^2 \tag{12}$$

$J_{(q,r)}$ denotes the Jacobian $[\partial(q,r)]/[\partial(x,z)]$. Note that the coordinate transformations familiar to compressible boundary-layer theory (e.g., the Dorodnitsyn transformation) cannot be used to reduce the system of equations to an incompressible form because their success depends on the normal pressure gradient being zero.

It can be shown that the self-similarity development in what follows for the special case of a linear dependance of density on temperature and with $\mu = k = 1$, also exists in the more general case with $\rho \sim T^{-1}$, $\mu \sim T^{\omega}$, and $k \sim T^{\Omega}$. However, inspection of the equations indicates that this generalization does not substantially alter our conclusions regarding the nature of the Boussinesq approximation, while substantially increasing the work involved in obtaining solutions.

We take $\mu = k = 1$ and also assume a linear variation of density with temperature

$$\rho = 1 + [(1/\rho_o)(d\rho^*/dt)]_o(t - t_o) = 1 - \alpha\theta T \tag{13}$$

where α is a dimensionless thermal expansion coefficient $(\alpha_o t_o)$ and θ a dimensionless driving potential

$$\theta = (t_w - t_o)/t_o \tag{14}$$

The form of Eqs. (13) and (14) can also be used to describe the case where density variations arise due to molecular diffusion driven by a concentration gradient. Equation (13) can be viewed as an approximation to the behavior of the fluid in a restricted temperature range. With these assumptions, Eqs. (11) and (12) assume the simplified form

$$-\delta\rho_x = \frac{\partial}{\partial z}\left\{\frac{1}{\rho}\left[-J_{(\psi_z,\psi)} + \psi_{zzz} - 2\rho_z\left(\frac{\psi_z}{\rho}\right)_z + \frac{1 - P}{P}\,\psi_z\,\frac{\rho_{zz}}{\rho}\right]\right\} \tag{15}$$

and

$$PJ_{(\rho,\psi)} = \rho_{zz} \tag{16}$$

The boundary conditions for the flow over a flat plate are

$$\begin{gathered} \psi_{(x,o)} = \psi_{z(x,o)} = \psi_{zz(x,\infty)} = 0, \qquad \rho_{(x,o)} = 1 - \alpha\theta, \\ \rho_{(x,\infty)} = 1 \qquad \psi_{z(x,\infty)} = u_e(x) \end{gathered} \tag{17}$$

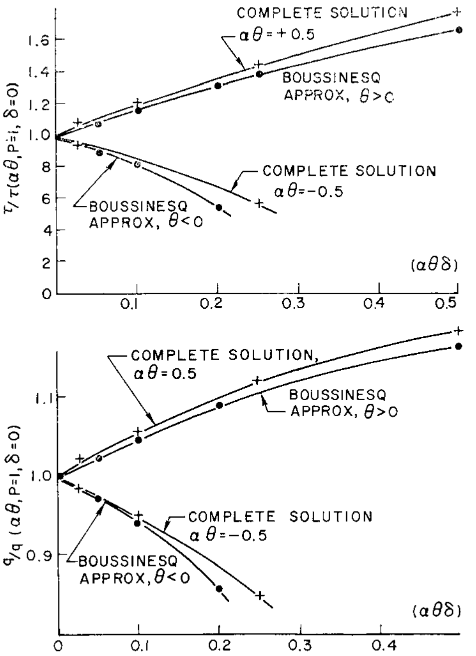

Fig. 1 The influence of buoyancy and the Boussinesq approximation on the wall shear stress and heat flux; $U_e = x^{1/5}$, $P = 1$, $\rho = 1 + \alpha\theta T$, $\mu = k = 1$.

These equations possess a similarity form given by

$$\psi = x^{3/5} f_{(\eta)} \tag{18a}$$

$$\rho = h_{(\eta)} \tag{18b}$$

$$\eta = z/x^{2/5} \tag{18c}$$

whereby they reduce to

$$f''' + \tfrac{3}{5} f f'' + \tfrac{1}{5}(h - f'^2) - 2h'\left(\frac{f'}{h}\right)' + \frac{1-P}{P} f' \frac{h''}{h} = \tfrac{2}{5}\delta h\left[\eta(1-h) + \int_\eta^\infty (1-h)d\eta\right] \tag{19}$$

and

$$h'' + \tfrac{3}{5} P f h' = 0 \tag{20}$$

Equation (19) has already been integrated once with respect to η to reduce it to a third-order equation. The boundary conditions (17) transform to

$$f_{(o)} = f_{(o)}' = f''_{(\infty)} = 0, \quad h_{(o)} = 1 - \alpha\theta, \quad h_{(\infty)} = 1, \tag{21}$$

$$f'_{(\infty)} = 1$$

where

$$u_e(x) = x^{1/5} \tag{22}$$

Thus, the similarity form applies only for a flow which is accelerated according to Eq. (22).

The Boussinesq approximation can be derived from the preceding set by means of a limit process (Mihaljan[4]). We make the substitutions

$$h_{(\eta)} = 1 - \alpha\theta T = 1 - \alpha\theta g_{(\eta)} \tag{23}$$

$$f_{(\eta)} = F_{(\eta)}$$

and take the limit of the resulting equations as $\alpha\theta$ vanishes with $\alpha\theta\delta$ fixed. It is easy to show that the limiting equations are

$$F''' + \tfrac{3}{5}FF'' + \tfrac{1}{5}(1 - F'^2) = \tfrac{2}{5}\alpha\theta\delta\left[\eta g + \int_\eta^\infty g d\eta\right] \tag{24}$$

$$g'' + \tfrac{3}{5}PFg' = 0 \tag{25}$$

$$F_{(o)} = F'_{(o)} = F''_{(\infty)} = 0, \quad F'_{(\infty)} = g_{(o)} = 1, g_{(\infty)} = 0 \tag{26}$$

Note that the Boussinesq system is characterized by only one parameter $(\alpha\theta\delta)$ while the non-Boussinesq system depends on the two parameters $\alpha\theta$ and δ separately.

Due to the different definitions of the similarity functions, the velocity components for the Boussinesq (B) and non-Boussinesq (NB) cases are given, respectively, by the expressions

$$u_B = x^{1/5} F'_{(\eta)} \tag{27a}$$

$$u_{NB} = x^{1/5}[f'_{(\eta)}/h_{(\eta)}] \tag{27b}$$

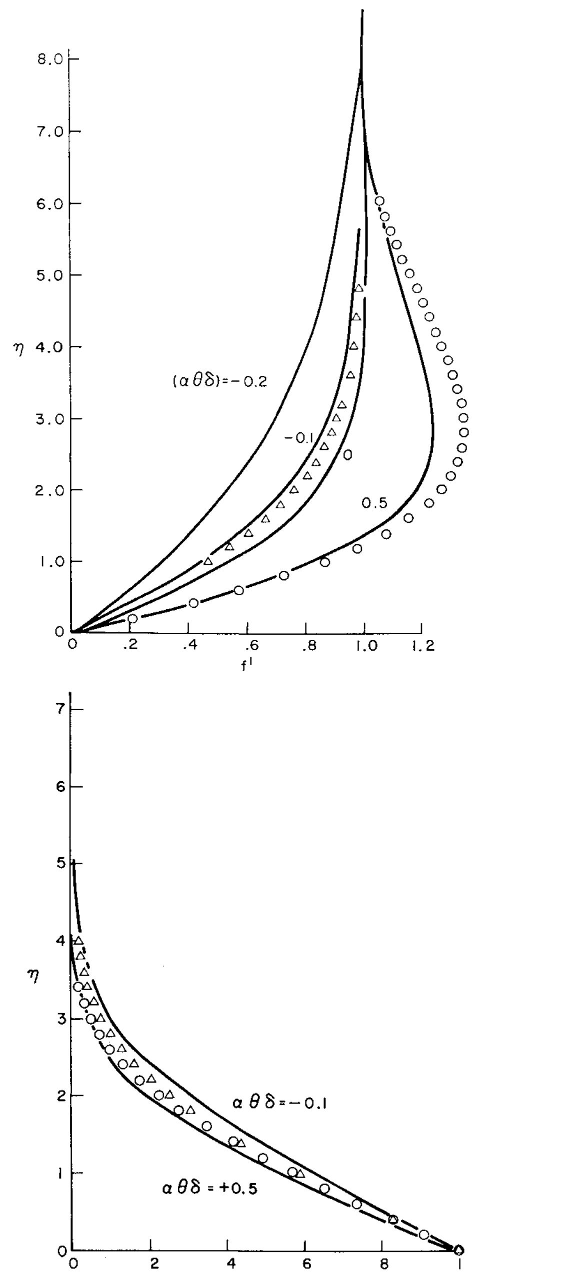

Fig. 2 Boundary-layer profiles of; a) velocity; and b) temperature for $U_e = x^{1/5}$, $P = 1$; (—Boussinesq approximation; o complete solution $\alpha\theta = \alpha\theta\delta = 0.5$; Δ complete solution $\alpha\theta = -0.5$, $\alpha\theta\delta = -0.1$).

and

$$w_B = -\tfrac{3}{5}z^{-2/5}(F - \tfrac{2}{3}\eta F') \quad (28a)$$

$$w_{NB} = -\tfrac{3}{5}x^{-2/5}1/h(f - \tfrac{2}{3}\eta f') \quad (28b)$$

The temperature distributions are determined from the relations

$$T_B = (t - t_0)/(t_w - t_0) = g_{(\eta)} \quad (29a)$$

$$T_{NB} = (t - t_0)/(t_w - t_0) = [1 - h_{(\eta)}]/\alpha\theta \quad (29b)$$

Similarly, expressions for the dimensionless skin-friction coefficients are

$$C_f{}^B = \tau_w/\rho_o U_o{}^2 = x^{-1/5}R^{-1/2}F''_{(o)} \quad (30a)$$

$$C_f{}^{NB} = x^{-1/5}R^{-1/2}f''_{(o)}/1 - \alpha\theta \quad (30b)$$

and the heat-transfer coefficients are given by

$$C_h{}^B = q_w/\rho_o c_{p_o} U_o(t_o - t_w) = (x^{-2/5}/PR^{1/2})g'_{(o)} \quad (31a)$$

$$C_h{}^{NB} = x^{-2/5}/PR^{1/2}[-(h'_{(o)}/\alpha\theta)] \quad (31b)$$

A result which helps in understanding the role of buoyancy in a horizontal boundary layer or shear layer can be derived using the vertical momentum Eq. (3) and the equation of state (13). One can show that

$$\partial\pi/\partial x = \partial\pi_\infty/\partial x - \alpha\theta\delta \frac{\partial}{\partial x}\int_z^\infty T dz \quad (32)$$

The first term on the right hand side denotes the freestream pressure gradient and the second term contains the effect of buoyancy. The integral is always positive, so the effect of buoyancy is equivalent to an induced longitudinal pressure gradient whose sign depends on the sign of θ. Buoyancy acts as an acceleration and increases the shear and heat transfer when θ is positive ($t_w > t_0$) and vice versa when θ is negative ($t_w < t_0$). This general result was also noted by Sparrow and Minkowycz.

Analysis for Arbitrary Buoyancy

The similarity solutions derived in the preceding section apply to one particular freestream acceleration. This arose due to the coupling between the streamwise variation of the temperature and the velocity via the baroclinic term in the vorticity equation. In this section we derive a similarity solution for general freestream acceleration by restricting the magnitude of the baroclinic term and using a perturbation expansion. The analysis is carried out only for the Boussinesq equations.

The Boussinesq form of the vorticity and energy equations (15) and (16) can be written as (taking $u = \psi_z$ and $w = -\psi_x$)

$$\partial/\partial z[J(\psi_z,\psi) - \psi_{zzz}] + \alpha\theta\delta T_x = 0 \quad (33)$$

and

$$PJ_{(T,\psi)} = T_{zz} \quad (34)$$

Table 1 Shear and heat-transfer coefficient, negligible buoyancy, $P = 1, \delta = 0, u_e = x^{1/5}, \rho = 1 + \alpha\theta T, \mu = k = 1$

$\alpha\theta$	$\tau = f''_{(o)}/(1 - \alpha\theta)$	$q = -h_{(o)}'/\alpha\theta$
0	0.621	−0.405
0.1	0.620	−0.390
0.3	0.618	−0.378
0.5	0.614	−0.360
0.75	0.614	−0.334
−0.3	0.631	−0.430
−0.5	0.635	−0.447
−0.75	0.638	−0.460

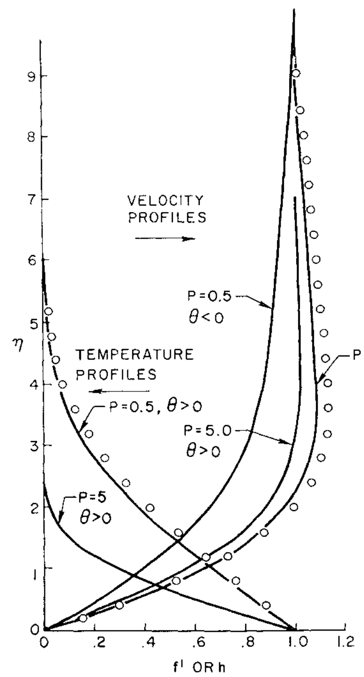

Fig. 3 The effect of Prandtl number on the velocity and temperature profiles for $U_e = x^{1/5}$ and $|\alpha\theta\delta| = 0.1$; (— Boussinesq approximation; o complete solution, $\alpha\theta = 0.5$).

with boundary conditions

$$\begin{gathered}\psi_{(x,o)} = \psi_{z(x,0)} = \psi_{zz(x,\infty)} = T_{x(\infty)} = 0 \\ T_{(x,0)} = 1, \quad \text{and} \quad \psi_{z(x,\infty)} = u_{e(x)}\end{gathered} \quad (35)$$

Restricting the buoyancy parameter $\alpha\theta\delta$ to be small compared to unity, the effect of buoyancy can be reasonably evaluated by a parameter expansion with $\alpha\theta\delta$ as the expansion parameter. Writing the expansions

$$\psi = \psi^{(1)} + \alpha\theta\delta\psi^{(2)} + \ldots = x^{(m+1)/2}g_{1(\eta)} + \alpha\theta\delta x^{1-2m}g_{2(\eta)} + \ldots \quad (36a)$$

and

$$T = T^{(1)} + \alpha\theta\delta T^{(2)} + \ldots = s_{1(\eta)} + \alpha\theta\delta x^{(1-5m)/2}s_{2(\eta)} + \ldots \quad (36b)$$

where

$$\eta = z/x^{(1-m)/2} \quad (37)$$

and substituting into Eqs. (33) and (34), the following sequence of equations result

$$g'''_1 + \frac{1+m}{2}g_1g''_1 + m(1 - g'_1{}^2) = 0, \quad g_{1(o)} = g'_{1(o)} = 0, \quad (38a)$$

$$g'_{1(\infty)} = 1$$

$$s''_1 + P\frac{m+1}{2}g_1s'_1 = 0, \quad s_{(o)} = 1, \quad s_{(\infty)} = 0 \quad (38b)$$

$$g'''_2 + \frac{1+m}{2}g_1g''_2 - \frac{1-m}{2}g'_1g'_2 + (1-2m)g''_1g_2 = -\frac{1-m}{2}\left[\eta s_1 + \int_\eta^\infty s_1 d\eta\right], \quad g_{2(o)} = g'_{2(o)} = g'_{2(\infty)} = 0 \quad (39a)$$

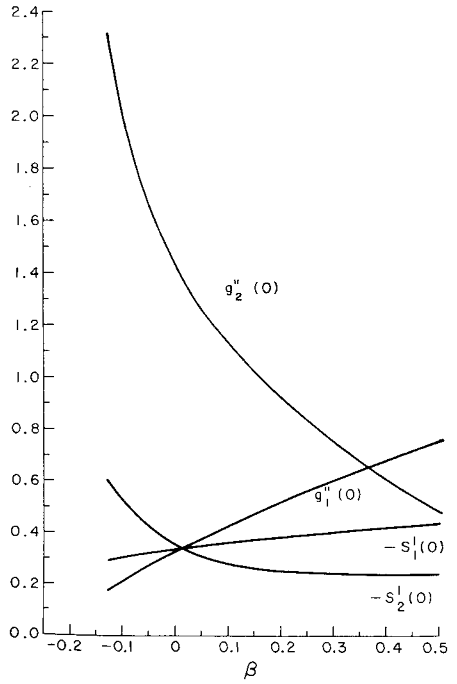

Fig. 4 Variation of the first- and second-order wall shear and heat flux with freestream acceleration.

and

$$s''_2 + P\left[\frac{1+m}{2}g_1s'_2 - \frac{1-5m}{2}g'_1s_2 + (1-2m)g_2s'_1\right] \tag{39b}$$
$$= 0,\ s_{2(o)} = s_{2(\infty)} = 0$$

The parameter m defines the acceleration of the inviscid freestream ($u_e = x^m$) and Eq. (38a) describes the well-known Falkner-Skan family of flows, usually characterized by the parameter $\beta = 2m/m + 1$ (cf., Rosenhead[5]). Mori[1] and later Sparrow and Minkowycz[2] solved the above set of equations with $m = 0$.

The coefficients of friction and heat transfer, defined earlier in Eqs. (30a) and (31a), are determined by the relations

$$C_f = \frac{x^{(3m-1)/2}}{R^{1/2}}g''_{1(o)}\left[1 + \alpha\theta\delta_x(1-5m)/2\,\frac{g''_{2(o)}}{g''_{1(o)}}\right] \tag{40a}$$

and

$$C_h = \frac{x^{m-1/2}}{PR^{1/2}}s'_{1(o)}\left[1 + \alpha\theta\delta x^{(1-5m)/2}\frac{s'_{2(o)}}{s'_{1(o)}}\right] \tag{40b}$$

Note that for the particular case $m = \frac{1}{5}$, the second-order or buoyancy contributions vary in the streamwise direction identically as the first-order terms and that their sign depends on the sign of θ.

Results

When buoyancy is negligible ($\delta \to 0$) the Boussinesq equations reduce to those for incompressible flow and become identical with one member of the Falkner-Skan family of self-similar flows. The solution of the full equations (NB) in the limit $\delta \to 0$ still retains the effect of the variation of the density and transport coefficients of the fluid on the viscous-inertia balance in the boundary layer. The solutions of the Boussinesq and the complete equations are identical only in the double limit $\delta \to 0$, $\alpha\theta \to 0$.

Table 1 lists the shear and heat-transfer coefficient calculated from the complete equations (NB) in the limit $\delta = 0$. The results represent the behavior of the boundary layer in a fluid having the assumed linear variation of properties with temperature. They compliment the already available tabulations for other fluids (e.g., Cohen and Reshotko[6]), for which $\rho = 1/T$, $\rho\mu = \rho k = 1$). The trends are well known.

Figure 1 compares the solutions of the complete equations and the Boussinesq approximation to these equations (also listed in Table 2). The latter is uniquely determined by the product $\alpha\theta\delta$; the former depends in addition on the parameter $\alpha\theta$. The comparison is made in terms of ratios of the shear and heat-transfer coefficients to their value in the limit $\delta = 0$. This emphasizes the effect of the buoyancy-inertia coupling which is neglected in the Boussinesq approximation. It is seen that the effect of buoyancy on the boundary layer is much larger than the error incurred in making the Boussinesq approximation, except at high-cooling rates where the flow tends to a buoyancy-induced separation.

A few typical profiles through the boundary layer are shown in Fig. 2. The curves represent the Boussinesq approximation and points give the relevant results of the complete (NB) solution. The Boussinesq approximation is seen to always underestimate the velocity. Its influence on the temperature field depends on the direction of the wall heat flux, overestimating the temperature when the wall is cooled and vice versa for a heated boundary. The differences, however, are seen to be small. Notable is the overshoot and the associated inflection in the velocity profiles which occur as the effect of buoyancy increases with $\alpha\theta\delta > 0$.

Figure 3 gives examples of the effect of the coupling between buoyancy and the shift of the thermal layer relative to the momentum boundary layer which is represented by the Prandtl number. The full Eq. (19) contain a term which vanishes for the case $P = 1$ (and in the Boussinesq approximation); it is the contribution of this term which we evaluate. The results show that the Boussinesq approximation renders the influence of buoyancy proper quite well throughout the range of Prandtl numbers studied. However it must be used in the form of a correction for buoyancy to calculations of ordinary (nonbuoyant) boundary layers which portray the proper density-temperature variation of the fluid.

Results for Small Buoyancy

We now consider the results obtained from the integration of Eqs. (38) and (39). A summary of the important numerical quantities for several values of the Prandtl number and acceleration parameter m [or, alternatively $\beta = 2m/(m+1)$] is given in Table 3. The results for $\beta = 0$ and variable Prandtl number at the bottom of the table are taken from the work of Sparrow and Minkowycz.[2] The dependence on β for a Prandtl number of unity is shown in Fig. 4. Figure 4 shows that the effect of buoyancy is substantially greater for flows with low or negative acceleration ($\beta < 0$) than for those with positive acceleration. The principal reason for this effect lies in the strong dependence of the magnitude of the buoyancy forces on the vertical scale of the boundary layer. Rewrit-

Table 2 Effect of buoyancy, Boussinesq approximation, $u_e = x^{1/5}$

P	$\alpha\theta\delta$	$F_{(o)} = \tau$	$g'_{(o)} = q$
1	0	0.621	−0.405
1	0.05	0.673	−0.415
1	0.10	0.722	−0.424
1	0.50	1.040	−0.476
1	−0.05	0.566	−0.394
1	−0.10	0.508	−0.382
1	−0.20	0.342	−0.340
1	−0.50	Separated	
0.5	0.1	0.756	−0.331
0.5	−0.1	0.452	−0.234
5	0.1	0.678	−0.745
5	−0.1	0.559	−0.700

Table 3 Solutions of self-similar flows for small buoyancy

m	β	P	$g''_{1(o)}$	$s'_{1(o)}$	$g''_{2(o)}$	$s'_{2(o}$	$\int_0^\infty s/_1 d\eta$	$\lvert\alpha t\delta\rvert_{\text{sep}}$
$-\frac{1}{17}$	$-\frac{1}{8}$	1	0.186	−0.291	2.287	−0.592	1.937	0.0803
0	0	1	0.332	−0.332	1.424	−0.344	1.721	0.235
$\frac{1}{15}$	$\frac{1}{8}$	1	0.44	−0.362	1.061	−0.277	1.590	0.422
$\frac{1}{9}$	$\frac{1}{5}$	1	0.512	−0.378	0.910	−0.258	1.526	0.563
$\frac{1}{5}$	$\frac{1}{3}$	1	0.621	−0.405	0.698	−0.241	1.428	0.890
$\frac{1}{3}$	$\frac{1}{2}$	1	0.757	−0.440	0.488	−0.239	1.320	1.55
$\frac{1}{5}$	$\frac{1}{3}$	0.5	0.621	−0.312	1.035	−0.268	1.872	0.600
$\frac{1}{5}$	$\frac{1}{3}$	1	0.621	−0.405	0.698	−0.241	1.428	0.890
$\frac{1}{5}$	$\frac{1}{3}$	5	0.621	−0.724	0.272	−0.172	0.790	2.28
0	0	0.01	0.332	−0.052	15.28	−0.303	11.92	0.0218
0	0	0.7	0.332	−0.293	1.722	−0.357	1.958	0.193
0	0	10	0.332	−0.728	0.385	−0.229	0.778	0.863

ing the parameter $\alpha\theta\delta$ as

$$\alpha\theta\delta = \frac{\alpha_0(t_w - t_o)g}{U_o^2}\frac{L}{R^{1/2}} \sim \frac{\alpha_o(t_w - t_o)}{U_o^2}\delta_{bl} \tag{41}$$

where δ_{bl} denotes the boundary-layer thickness, demonstrates this dependence explicitly, as does the integral of the first-order temperature distribution displayed in the second to the last column of Table 3. Therefore, a decrease in either β, P, or both will increase the role of buoyancy forces in a horizontal boundary layer.

Another important result pertaining to the range of validity of the perturbation expansion (36) can be obtained by comparing the solution of the Boussinesq Eqs. (24) and (25) with the solution of the perturbation Eqs. (38) and (39) for $m = \frac{1}{5}$. It is interesting to note that the perturbation expansion underestimates the wall shear when the wall is heated and overestimates it when the wall is cooled. The direction is reversed for the heat flux. The error incurred by using the perturbation expansion rather than the complete equations is approximately ten percent for $\alpha\theta\delta = 0.5$, indicating that the results of Table 3 and Eqs. (40) should give reasonably accurate estimates for other freestream accelerations when the buoyancy parameter is less than one-half.

The last column in Table 3 gives the magnitude of the buoyancy parameter $\alpha\theta\delta$ for a cooled boundary ($\theta < 0$) required to induce separation. These values were calculated using Eq. (40a) with a dimensionless streamwise length of unity ($x = 1$). We observe that the magnitude of $\alpha\theta\delta$ required to achieve separation increases with increasing acceleration (β) and Prandtl number (P). This behaviour is not surprising when interpreted in the light of Eq. (41). We emphasize that these values are only rough estimates of the trend because a linear approximation to the variation of C_f and C_h with $\alpha\theta\delta$ as separation is approached (cf., Fig. 1) is obviously poor. The numerical integration of the equations as separation is approached became increasingly difficult and for this reason the curves of Fig. 1 were not extended to separation. However, by inspection of the figure we see that separation in the case $u_e = x^{1/5}$ would occur near $\alpha\theta\delta$ of 0.3 (Boussinesq case) whereas the perturbation solution predicts it at $\alpha\theta\delta = 0.890$.

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